QN.→ State and Prove Cauchy's integral formula for the derivative of an analytic function.

Ans.→ **Statement**:- Let $f(z)$ be analytic within and on the boundary C of a simply connected region D and let z_0 be any Point within C, then,

$$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\,dz.$$

Proof:- Let z_0+h be a Point in the neighbourhood of the Point z_0, then by Cauchy's integral formula, we have.

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz$$

and $$f(z_0+h) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0-h}\,dz$$

$$\therefore f(z_0+h) - f(z_0) = \frac{1}{2\pi i}\int_C \left[\frac{1}{z-z_0-h} - \frac{1}{z-a}\right] f(z)\,dz$$

$$= \frac{1}{2\pi i}\int_C \frac{h}{(z-z_0-h)(z-z_0)} f(z)\,dz$$

or, $$\frac{f(z_0+h) - f(z_0)}{h} = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0-h)(z-z_0)}$$

~~Hence,~~ When $h \to 0$

$$\therefore \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} = \lim_{h\to 0} \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)(z-z_0-h)}$$

$$\text{Or, } f'(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\,dz.$$

Proved.

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Higher order derivatives:-

QNo.→ State and Prove Analytic character of the Successive derivatives of an analytic function.

Ans. Statement:- Let $f(z)$ be analytic within and on the boundary C of a Simply Connected region D and let z_0 be any Point within C. Then $f(z)$ Possesses derivatives of all orders and these derivatives are themselves all analytic at z_0 their values being given by

$$f^n(z_0) = \frac{\lfloor n}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}$$

Proof:- Assume that the theorem is true for $n=m$

$$\text{i.e. } f^m(z_0) = \frac{\lfloor m}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{m+1}} \text{ is true.}$$

So that,

$$\frac{f^m(z_0+h)-f^m(z_0)}{h}$$

$$= \frac{1}{h}\cdot\frac{\lfloor m}{2\pi i}\left[\int_C \frac{f(z)\,dz}{(z-z_0-h)^{m+1}} - \int_C \frac{f(z)\,dz}{(z-z_0)^{m+1}}\right.$$

$$= \frac{1}{h} \cdot \frac{\lfloor m}{2\pi i} \int_C \left[\frac{1}{(z-z_0)^{m+1}} \left\{ \left(1 - \frac{h}{z-z_0} \right)^{-(m+1)} - 1 \right\} \right] f(z)\, dz$$

$$= \frac{1}{h} \cdot \frac{\lfloor m}{2\pi i} \int_C \left[\frac{1}{(z-z_0)^{m+1}} \left\{ (m+1) \frac{h}{z-z_0} + \frac{(m+1)(m+2)}{\lfloor 2} \frac{h^2}{(z-z_0)^2} \right. \right.$$

$$\left. \left. + \ldots \text{ terms with higher powers of } h \right\} \right] f(z)\, dz$$

Taking limit as $h \to 0$, we have

$$\underset{h\to 0}{Lt.} \frac{f^m(z_0+h) - f^m(z_0)}{h} = \frac{\lfloor (m+1)}{2\pi i} \int_C \left[\frac{f(z)\, dz}{(z-z_0)^{m+2}} \right]$$

$$\text{i.e. } f^{m+1}(z_0) = \frac{\lfloor (m+1)}{2\pi i} \int_C \frac{f(z)\, dz}{(z-z_0)^{m+2}}$$

Which shows that theorem is also true for $n = m+1$.

But we know that the theorem is true for $n=1$, therefore must be true for $n=2$ and so on

Hence it must be true for any +ve values of n.

$$\text{Thus, } f^n(a) = \frac{\lfloor n}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}}$$

Also, we see from this result that $f^n(a)$ is an analytic function of z: This implies that derivatives of $f(z)$ of all orders are analytic if $f(z)$ is analytic.

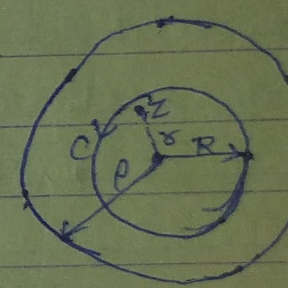
Q.No.→ State and Prove Poisson's integral formula for a Circle.

Ans. Statement:- Let $f(z)$ be analytic in the region $|z| < \rho$ and $z = re^{i\theta}$ be any Point of this region, then

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\, f(Re^{i\phi})}{R^2 - 2Rr\cos(\theta - \phi) + r^2}\, d\phi.$$

Where R is any number such that $0 < R < \rho$.

Proof:- Let C denote the Circle $|z| = R$, where $r < R < \rho$. Let z be any Point within C such that the distance z from the Centre R. Now, by Cauchy's integral formula, we have



$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(W)}{W - z}\, dW \quad \text{— (1)}$$

Clearly, the inverse of z with respect to the Circle C will be Point $\frac{R^2}{\bar{z}}$ and lies outside C. Hence the function $\frac{f(W)}{W - \frac{R^2}{\bar{z}}}$ and on C.

Hence, by Cauchy-Goursat theorem,

$$\int_C \frac{f(W)}{W - \frac{R^2}{\bar{z}}}\, dW = 0$$

$$\therefore \frac{1}{2\pi i}\int_C \frac{f(w)}{w-\frac{R^2}{\bar{z}}}\,dw = 0 \quad \text{--- (2)}$$

From, (1) − (2), we get

$$f(z) = \frac{1}{2\pi i}\int_C \left\{\frac{1}{w-z} - \frac{1}{w-R^2/\bar{z}}\right\} f(w)\,dw$$

$$= \frac{1}{2\pi i}\int_C \frac{z-R^2/\bar{z}}{(w-z)(w-R^2/\bar{z})} f(w)\,dw \quad \text{--- (3)}$$

We now, put $z = re^{i\theta}$ & $w = Re^{i\phi}$ $\therefore \bar{z} = re^{-i\theta}$

& $dw = Rie^{i\phi}\,d\phi$ ~~in~~ (3)

$$\therefore f(re^{i\theta}) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{\left(re^{i\theta} - \frac{R^2}{re^{-i\theta}}\right) f(Re^{i\phi})}{(Re^{i\phi}-re^{i\theta})\left(Re^{i\phi} - \frac{R^2}{re^{-i\theta}}\right)} \cdot Rie^{i\phi}\,d\phi$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{(r^2-R^2)e^{i\theta}\cdot Rie^{i\phi}\cdot d\phi\, f(Re^{i\phi})}{(Re^{i\phi}-re^{i\theta})(Rre^{i\phi}-R^2e^{i\theta})}\,d\phi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{(r^2-R^2)\, f(Re^{i\phi})}{(Re^{i\phi}-re^{i\theta})(re^{-i\theta}-Re^{-i\phi})}\,d\phi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)\, f(Re^{i\phi})}{(Re^{i\phi}-re^{i\theta})(Re^{-i\phi}-re^{-i\theta})}\,d\phi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)\, f(Re^{i\phi})}{R^2 - Rr\{e^{i(\theta-\phi)}+e^{-i(\theta-\phi)}\}+r^2}\,d\phi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)\, f(Re^{i\phi})}{R^2 - 2Rr\,e^{i(\theta-\phi)}+r^2}$$

i.e. $\frac{1}{2\pi}\int_0^{2\pi}\frac{(R^2-r^2)f(Re^{i\phi})\,d\phi}{R^2-2Rr\cos(\theta-\phi)+r^2}$ Proved.